

THE DOLD-KAN CORRESPONDENCE AND COALGEBRA STRUCTURES

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ABSTRACT. By using the Dold-Kan correspondence we construct Quillen adjunctions between the model categories of simplicial coalgebras and differential graded coalgebras. We also investigate the question whether these Quillen adjunctions can be improved to Quillen equivalences. We restrict to categories of connected coalgebras and prove a Quillen equivalence between them.

1. INTRODUCTION

The Dold-Kan correspondence establishes an equivalence between the category of simplicial objects and the category of non-negatively graded differential objects in abelian categories. In this paper we work with the abelian category of vector spaces over a fixed field K . We investigate the model categorical properties of the functors involved in the Dold-Kan correspondence on the level of the corresponding model categories of comonoids. We show that the Dold-Kan correspondence gives rise to Quillen adjunctions between the model categories of simplicial and non-negatively graded differential counital coalgebras. We investigate some approaches to extending the resulting adjunctions to an equivalence on the level of homotopy categories.

Section 2 is devoted to some generalities about the monoidal categories of non-negatively graded differential vector spaces, simplicial vector spaces and their corresponding categories of comonoids.

Section 3 is a dualization of the work by Schwede and Shipley in [SS03]. First, we show that the normalization functor N defines a functor from the category of simplicial coalgebras to the category of non-negatively differential graded coalgebras. Similar considerations hold for the functor Γ , however, as in [SS03], we point out that both functors are neither adjoint nor inverse to each other on the level of comonoids. Then, we construct functors that are right adjoint to the functors N and Γ on the level of comonoids. The adjoint pairs of functors obtained in this way turn out to be adjoint Quillen pairs. Finally, we discuss whether these Quillen adjunctions can be improved to Quillen equivalences via a categorical dualization of a result by Schwede-Shipley and a criterion by Hovey.

Section 4 restricts to the categories of connected simplicial coalgebras and connected differential graded coalgebras. We investigate the completeness and cocompleteness properties of connected coalgebras. With these properties, we derive from [GG99], [Smi11] and [Goe95] model category structures for categories of connected coalgebras. We prove a Quillen equivalence between the categories of connected differential graded and simplicial coalgebras.

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Section 5 closes this paper with an appendix on the category of connected differential graded algebras, which is helpful for constructing limits for connected differential graded coalgebras.

2. PRELIMINARIES

2.1. Differential graded vector spaces. Let \mathbf{Vct} be the category of vector spaces over a fixed field K . We denote by \mathbf{DGVct} the category of differential graded K -vector spaces which are concentrated in non-negative degrees and have differentials of degree -1 . We denote by \mathbb{S}^n the n -sphere chain complex. This is the object of \mathbf{DGVct} which has the field K in degree n and 0 in other degrees. All differentials in \mathbb{S}^n are trivial. Secondly, the n -disk, denoted by \mathbb{D}^n , is the object of \mathbf{DGVct} which has the field K in degrees n and $n-1$ and 0 elsewhere. The identity on K is its only non-trivial differential.

Recall that if X and Y are two objects in the category \mathbf{DGVct} , the monoidal product \otimes is given by $(X \otimes Y)_n = \bigoplus_{p+q=n} X_p \otimes_K Y_q$ with differential $d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$. The unit of this monoidal product is \mathbb{S}^0 , the differential graded vector space concentrated in degree 0 which we sometimes denote by $K[0]$. Notice that the monoidal product \otimes is symmetric. The category \mathbf{DGVct} endowed with the monoidal product \otimes is closed. Given differential graded vector spaces (X, d_X) and (Y, d_Y) , let $\text{Hom}(X, Y)$ be the object of \mathbf{DGVct} with:

$$\text{Hom}(X, Y)_n = \prod_{p \geq 0} \mathbf{Vct}(X_p, Y_{p+n})$$

with differential d_H for any map $f = \{f_p: X_p \rightarrow Y_{p+n}\}_{p \geq 0}$ given by

$$(d_H f)_p(x) = d_Y(f_p x) + (-1)^{n+1} f_{p-1}(d_X x), \quad x \in X_p.$$

The specified right adjoint of the functor $- \otimes Y: \mathbf{DGVct} \rightarrow \mathbf{DGVct}$ is then given by the functor $\text{Hom}(Y, -): \mathbf{DGVct} \rightarrow \mathbf{DGVct}$.

2.2. Simplicial vector spaces. We denote by \mathbf{SVct} the category of simplicial vector spaces, that is, a functor $X: \Delta^{\text{op}} \rightarrow \mathbf{Vct}$, where Δ is the category of finite ordered sets.

Let X and Y be two objects in \mathbf{SVct} . The monoidal product $\widehat{\otimes}$ is given by $(X \widehat{\otimes} Y)_n = X_n \otimes_K Y_n$ with coordinatewise structure maps. The unit of the monoidal product $\widehat{\otimes}$ is the simplicial vector space that has K in each degree and identity maps on K as face and degeneracy operators. We denote this unit by $I(K)$. This monoidal product $\widehat{\otimes}$ is symmetric and \mathbf{SVct} is closed.

2.3. Differential graded coalgebras. We denote by $\mathbf{DGcoAlg}$ the category of counital coassociative differential graded K -coalgebras. In other words $\mathbf{DGcoAlg}$ is the category of comonoids in the monoidal category $(\mathbf{DGVct}, \otimes, K[0])$. In the coming sections the symbols \square and \sqcup stand respectively for the categorical product and coproduct in $\mathbf{DGcoAlg}$.

Lemma 2.1. [GG99, Proposition 1.10]. *The forgetful functor U_d from the category of differential graded coalgebras to the category of differential graded vector spaces has a right adjoint S_d .*

2.4. Simplicial coalgebras. The category of simplicial coalgebras, denoted by \mathbf{ScoAlg} , is the category of comonoids in the monoidal category $(\mathbf{SVct}, \widehat{\otimes}, I(K))$.

Lemma 2.2. *The forgetful functor U_s from the category of simplicial coalgebras to the category of simplicial vector spaces has a right adjoint S_s .*

Here, the functor S_s is obtained by extending degreewise the cofree coalgebra functor S from the category of vector spaces to the category of coalgebras as constructed in [Swe69, Theorem 6.4.1].

3. A COMPARISON OF COALGEBRAS CATEGORIES

3.1. Dold-Kan functors for coalgebras. The Dold-Kan correspondence asserts that **SVct** and **DGVct** are essentially the same. This equivalence of categories is achieved by the normalization functor N and its inverse Γ . We refer to [Wei94, 8.8.4] for a detailed description of these functors. Moreover, it is well-known that the normalized versions of the Alexander-Whitney map $AW: N(A \widehat{\otimes} B) \rightarrow NA \otimes NB$ and of the shuffle map $\nabla: NA \otimes NB \rightarrow N(A \widehat{\otimes} B)$ both turn the normalization functor into a monoidal functor. From these observations, we derive that the normalization functor N and its inverse Γ pass on the level of comonoids as stated in propositions below.

Proposition 3.1. *If $(A, \Delta_A, \varepsilon_A)$ is a simplicial coalgebra, then $(NA, \Delta_{NA}, \varepsilon_{NA})$ is a differential graded coalgebra with a comultiplication given by the composition*

$$NA \xrightarrow{N(\Delta_A)} N(A \widehat{\otimes} A) \xrightarrow{AW_{A,A}} NA \otimes NA$$

and counit given by $N(\varepsilon_A)$.

Proposition 3.2. *If $(B, \Delta_B, \varepsilon_B)$ is a differential graded coalgebra, then $(\Gamma B, \Delta_{\Gamma B}, \varepsilon_{\Gamma B})$ is a simplicial coalgebra with a comultiplication $\Delta_{\Gamma B}$ given by the following composition*

$$\begin{array}{ccccc} \Gamma B & \xrightarrow{\Gamma(\Delta_B)} & \Gamma(B \otimes B) & \xrightarrow{\Gamma(\varepsilon_B^{-1} \otimes \varepsilon_B^{-1})} & \Gamma(N\Gamma B \otimes N\Gamma B) \\ & \searrow \Delta_{\Gamma B} & & & \downarrow \Gamma(\nabla_{\Gamma B, \Gamma B}) \\ & & & & \Gamma N(\Gamma B \widehat{\otimes} \Gamma B) \\ & & & & \downarrow \eta_{\Gamma B \widehat{\otimes} \Gamma B}^{-1} \\ & & & & \Gamma B \widehat{\otimes} \Gamma B \end{array}$$

and counit given by $\Gamma(\varepsilon_B)$.

Notice that both propositions and their proofs are dual to results in [SS03, Section 2.3]. We refer to [Sor10, Section 5.1] for detailed proofs.

Dually to [SS03, Section 2.4], we point out that the coalgebra-valued functors N and Γ are neither adjoint nor inverse to each other. This fact is specified in the remark below.

Lemma 3.3. *The adjunction counit $\varepsilon: N\Gamma \rightarrow \text{Id}$ is a comonoidal transformation. Let $\psi_{X,Y}$ be the composition of natural maps*

$$\psi_{X,Y} = \eta_{\Gamma X \widehat{\otimes} \Gamma Y}^{-1} \circ \Gamma(\nabla_{\Gamma X, \Gamma Y}) \circ \Gamma(\varepsilon_X^{-1} \otimes \varepsilon_Y^{-1}).$$

Then the diagram

$$\begin{array}{ccccc} N\Gamma(X \otimes Y) & \xrightarrow{N(\psi_{X,Y})} & N(\Gamma X \widehat{\otimes} \Gamma Y) & \xrightarrow{AW_{\Gamma X, \Gamma Y}} & N\Gamma X \otimes N\Gamma Y \\ \varepsilon_{X \otimes Y} \downarrow & & & & \downarrow \varepsilon_X \otimes \varepsilon_Y \\ X \otimes Y & \xlongequal{\quad} & X \otimes Y & \xlongequal{\quad} & X \otimes Y \end{array}$$

commutes for every X, Y in **DGcoAlg**.

Proposition 3.4. *The functor $\Gamma: \mathbf{DGcoAlg} \rightarrow \mathbf{DGcoAlg}$ is full and faithful and respects coalgebra structures. Moreover, the composite endofunctor $N\Gamma$ is naturally isomorphic to the identity functor on the level of categories of comonoids.*

Remark 3.5. The unit $\eta: \text{Id} \rightarrow \Gamma N$ does not have good comonoidal properties. More precisely, there are objects X and Y in the category **ScoAlg** such that the diagram

$$\begin{array}{ccc} X \widehat{\otimes} Y & \xlongequal{\quad} & X \widehat{\otimes} Y \\ \downarrow \eta_{X \widehat{\otimes} Y} & & \downarrow \eta_{X \widehat{\otimes} Y} \\ \Gamma N(X \widehat{\otimes} Y) & \xrightarrow{\Gamma(AW_{X,Y})} & \Gamma(NX \otimes NY) \xrightarrow{\psi_{NX,NY}} \Gamma NX \widehat{\otimes} \Gamma NY \end{array}$$

does not commute. Indeed, consider for example $X = Y = \Gamma(\mathbf{Z}[1])$ as in [SS03, Remark 2.14]. Since N is left inverse to Γ by the previous proposition, one has

$$NX = NY = N\Gamma(\mathbf{Z}[1]) \cong \mathbf{Z}[1] \text{ and } NX \otimes NY \cong \mathbf{Z}[1] \otimes \mathbf{Z}[1] = \mathbf{Z}[2].$$

Therefore the lower composite map in the previous diagram vanishes in degree 1 since

$$[\Gamma(NX \otimes NY)]_1 = [\Gamma(\mathbf{Z}[2])]_1 = 0.$$

But in degree 1, the right map $\eta_{X \widehat{\otimes} Y}$ is an isomorphism between free abelian groups of rank one since

$$[\Gamma(NY)]_1 \cong [Y]_1 \cong \mathbf{Z} \cong [X]_1 \cong [\Gamma(NX)]_1.$$

3.2. Model category structures. Now, we consider Quillen's setting of model category and we investigate whether the coalgebra-valued functors N and Γ fit under this framework. Before, we recall the model category structures of the various categories involved in this work.

3.2.1. Model structure of \mathbf{DGVct} . The category **DGVct** is equipped with a cofibrantly generated model structure proven in [Qui67, Chapter I, Example B] or in [DS95, Chapter 7]. A map f in the category **DGVct** is a weak equivalence if $H_* f$ is an isomorphism, a cofibration if for each $n \geq 0$, f_n is injective, and a fibration if for each $n \geq 1$, f_n is surjective. The generating acyclic cofibrations are given by the maps $\{0 \rightarrow \mathbb{D}^n \mid n \geq 1\}$ and the generating cofibrations by $\{\mathbb{S}^{n-1} \rightarrow \mathbb{D}^n \mid n \geq 1\}$.

3.2.2. Model structure of $\mathbf{DGcoAlg}$. The category **DGcoAlg** is endowed with a model structure in [GG99, Section 2]. A map f in the category **DGcoAlg** is a weak equivalence if $H_* f$ is an isomorphism, a cofibration if it is a degreewise injection of graded K -vector spaces, and a fibration if it has the right lifting property with respect to trivial cofibrations.

3.2.3. Model structure of \mathbf{SVct} . The category of simplicial vector spaces has a cofibrantly generated model structure proven in [Qui67, II.4, II.6]. The model structure comes from defining a map f in the category **SVct** to be a weak equivalence if $\pi_* f \cong H_* N f$ is an isomorphism, a cofibration if it is a level-wise injection, and a fibration if it has the right lifting property with respect to trivial cofibrations. Since **DGVct** is cofibrantly generated, one can deduce that **SVct** is cofibrantly generated by applying the transfer result by Crans in [Cra95, Section 3] to the adjoint pair (Γ, N) provided by the Dold-Kan correspondence. Hence in the category **SVct**, the generating acyclic cofibrations are given by the maps $\{0 \rightarrow \Gamma(\mathbb{D}^n) \mid n \geq 1\}$ and the generating cofibrations by $\{\Gamma(\mathbb{S}^{n-1}) \rightarrow \Gamma(\mathbb{D}^n) \mid n \geq 1\}$.

3.2.4. Model structure of \mathbf{ScoAlg} . The category of cocommutative simplicial coalgebras has a model structure proven in [Goe95, Section 3]. But one can adapt the arguments therein to the non-cocommutative case as well. A map f in \mathbf{ScoAlg} is a weak equivalence if $\pi_* f \cong H_* N f$ is an isomorphism, a cofibration if it is a levelwise inclusion, and a fibration if it has the right lifting property with respect to trivial cofibrations.

Goerss' main line of argumentation remains unchanged for the case of non cocommutative simplicial coalgebras. The only slight difference is concerned with the lemma recalled below.

Lemma 3.6. [Goe95, Lemma 3.5]. *Let $f: C \rightarrow D$ be a morphism of coalgebras. Then, f can be factored as $f = p \circ i$*

$$C \xrightarrow{i} X \xrightarrow{p} D$$

where i is a cofibration and p is an acyclic fibration.

In the proof of this lemma we may replace the cocommutative cofree functor by its non-cocommutative version $S_s: \mathbf{SVct} \rightarrow \mathbf{ScoAlg}$. In this way, the arguments of Goerss transfer to the non-cocommutative setting since only the cofreeness property is required.

3.3. Quillen adjunctions for coalgebras. We compare \mathbf{ScoAlg} and $\mathbf{DGcoAlg}$ in terms of Quillen setting. The situation may be illustrated in the diagram

$$\begin{array}{ccc} (\mathbf{SVct}, \widehat{\otimes}, I(K)) & \begin{array}{c} \xleftarrow[N]{\Gamma} \\ \xrightarrow[\widetilde{N}]{\Gamma} \end{array} & (\mathbf{DGVct}, \otimes, K[0]) \\ U_s \uparrow\downarrow S_s & & U_d \uparrow\downarrow S_d \\ \mathbf{ScoAlg} & \xrightarrow[\widetilde{N}]{\Gamma} & \mathbf{DGcoAlg}. \end{array}$$

where \widetilde{N} stands for the coalgebra-valued normalization functor.

Proposition 3.7. *In the above situation the functor \widetilde{N} has a right adjoint R^{com} . Moreover the adjoint pair $(\widetilde{N}, R^{\text{com}})$ is a Quillen pair.*

Proof. First, let V be a differential graded vector space and $S_d(V)$ its differential graded cofree coalgebra. We set

$$R^{\text{com}}(S_d(V)) = S_s \Gamma(V).$$

Indeed, the various adjoint pairs (U_s, S_s) , (N, Γ) , (U_d, S_d) and the identity $N U_s = U_d \widetilde{N}$ respectively, yield the following bijections

$$\begin{aligned} \mathbf{ScoAlg}(X, R^{\text{com}} S_d V) &= \mathbf{ScoAlg}(X, S_s \Gamma V) \\ &\cong \mathbf{SVct}(U_s X, \Gamma V) \\ &\cong \mathbf{DGVct}(N U_s X, V) \\ &\cong \mathbf{DGVct}(U_d \widetilde{N} X, V) \\ &\cong \mathbf{DGcoAlg}(\widetilde{N} X, S_d V). \end{aligned}$$

This implies that the functor R^{com} is right adjoint to \widetilde{N} for cofree coalgebras. Now we notice that the adjunction (U_d, S_d) defines a monad $S_d U_d$ over the category $\mathbf{DGcoAlg}$. Thus, if C is a differential graded coalgebra, it can be written as the equalizer of the diagram

$$S_d U_d C \rightrightarrows_{d^1}^{d^0} S_d U_d S_d U_d C$$

Since the functor R^{com} should be a right adjoint it has to preserve limits. Therefore defining $R^{\text{com}}(C)$ as the equalizer of the maps $R^{\text{com}}(d^0)$ and $R^{\text{com}}(d^1)$ yields the desired right adjoint.

Finally we observe that the cofibrations and acyclic cofibrations in **SVct** and **DGVct** match with those of their respective categories of comonoids **ScoAlg** and **DGcoAlg**. Since the functor N is a left Quillen functor the identity $NU_s = U_d\tilde{N}$ ensures that the functor \tilde{N} is a left Quillen functor. \square

Remark 3.8. We mention that instead of \tilde{N} , we could consider the coalgebra-valued functor $\tilde{\Gamma}$. By similar techniques, it is possible to construct a right adjoint functor R^{com} to $\tilde{\Gamma}$ and show that $(\tilde{\Gamma}, R^{\text{com}})$ is a Quillen pair.

3.4. Are **ScoAlg and **DGcoAlg** Quillen equivalent?** We investigate whether **ScoAlg** and **DGcoAlg** are Quillen equivalent via the Quillen pair (N, R^{com}) . On one hand we try a categorical dualization of a result by Schwedey and Shipley in [SS03] and on the other hand we check a criterion by Hovey in [Hov99].

3.4.1. Schwede-Shipley theorem. In order to improve the pair $(\tilde{N}, R^{\text{com}})$ to a Quillen equivalence, we dualize and check required conditions in [SS03, Theorem 3.12.(3)]. In the propositions below, we find good candidates for, respectively, generating acyclic fibrations and generating fibrations in **DGVct**.

Proposition 3.9. Define Q to be the set $\{\mathbb{D}^n \rightarrow 0 \mid n \geq 1\}$. A chain map $f: X \rightarrow Y$ is injective if and only if f has the left lifting property with respect to any map in Q .

Proof. One direction is obvious since maps $\mathbb{D}^n \rightarrow 0$ are acyclic fibrations in the standard model structure of **DGVct**.

For the other direction, suppose f has the left lifting property with respect to Q . Assume that there is an $x_n \in X_n$ with $x_n \neq 0$ and $f_n(x_n) = 0$. Since $x_n \neq 0$ there is a linear map $\alpha_n: X_n \rightarrow K$ such that $\alpha_n(x_n) = 1$.

- (i) If x_n is not a cycle then define $\alpha_{n-1}(dx_n) = 1$. This gives a chain map $\alpha: X \rightarrow \mathbb{D}^n$ and by assumption a lift ζ exists in the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \mathbb{D}^n \\ f \downarrow & \swarrow \zeta & \downarrow \sim \\ Y & \xrightarrow{\beta} & 0 \end{array}$$

Therefore $0 = \zeta_n(0) = \zeta_n \circ f_n(x_n) = \alpha_n(x_n) = 1$, which is the desired contradiction. Therefore $f_n(x_n) \neq 0$ and the chain map f is injective.

- (ii) Let x_n be a cycle and a boundary. That is, $d(x_n) = 0$ and there is an $x_{n+1} \in X_{n+1}$ such that $d(x_{n+1}) = x_n$. Then define $\alpha_{n+1}(x_{n+1}) = 1$. Note that x_{n+1} is not a boundary. This gives a chain map $\alpha: X \rightarrow \mathbb{D}^{n+1}$ and a contradiction follows as in case (i).
- (iii) Let x_n be a cycle and not a boundary. As we are working with field coefficients, x_n then generates a sphere sub-complex \mathbb{S}^n of X . This sub-complex \mathbb{S}^n can be mapped to \mathbb{D}^{n+1} and the above argument works again.

\square

Proposition 3.10. Define P to be the set $\{\mathbb{D}^n \rightarrow 0, \mathbb{D}^n \rightarrow \mathbb{S}^n \mid n \geq 1\}$. A chain map f is in P -proj if and only if it is injective and induces an isomorphism in homology.

Proof. One direction is obvious since every map in P is a fibration. For the other direction, injectivity comes from $P\text{-proj} \subset Q\text{-proj}$ (since $Q \subset P$) and the previous lemma. We show that every map in $P\text{-proj}$ induces isomorphism in homology. Let $f: A \rightarrow B$ be a map in $P\text{-proj}$. As f is injective, we have the short exact sequence $0 \rightarrow A \rightarrow B \rightarrow \text{coker } f \rightarrow 0$. As this sequence induces a long exact sequence in homology, it suffices to show that $\text{coker } f$ has no homology. The object $\text{coker } f$ which is built degreewise is also a pushout

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ f \downarrow & & \downarrow \\ B & \longrightarrow & \text{coker } f \end{array}$$

We deduce that the map $0 \rightarrow \text{coker } f$ is also in $P\text{-proj}$ since $P\text{-proj}$ is closed under pushouts. Let us now assume that $\text{coker } f$ is *not* acyclic. Then there is an element $x_n \in (\text{coker } f)_n$ with $d(x_n) = 0$ and x_n not a boundary. Note that x_n generates a subcomplex \mathbb{S}^n of $\text{coker } f$ and that $\text{coker } f \cong \mathbb{S}^n \oplus \text{coker } f / \mathbb{S}^n$ as chain complexes. But the map $0 \rightarrow \mathbb{S}^n$ is not in $P\text{-proj}$ since

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{D}^n \\ \downarrow & & \downarrow \\ \mathbb{S}^n & \xlongequal{\quad} & \mathbb{S}^n \end{array}$$

does not admit a lift. Therefore the map $0 \rightarrow \text{coker } f$ cannot be in $P\text{-proj}$ either which is the desired contradiction. \square

Unfortunately, the codomains \mathbb{S}^n in the set P are not cosmall. Indeed, consider a cofiltered sequence of maps in the category **DGVct**

$$\cdots \rightarrow Y_{\alpha+1} \rightarrow Y_\alpha \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0$$

such that each $Y_{\alpha+1} \rightarrow Y_\alpha$ is in P -cocell. Then the following canonical map

$$\operatorname{colim}_\alpha \mathbf{DGVct}(Y_\alpha, \mathbb{S}^n) \longrightarrow \mathbf{DGVct}(\lim_\alpha Y_\alpha, \mathbb{S}^n)$$

does not need to be a bijection: for instance, consider Y_α to be \mathbb{S}^n and all maps $Y_\alpha \rightarrow Y_{\alpha-1}$ to be trivial.

Moreover, we lose closedness property of the category $((\mathbf{DGVct})^\text{op}, \otimes)$ since the functor $- \otimes Y: (\mathbf{DGVct})^\text{op} \rightarrow (\mathbf{DGVct})^\text{op}$ does not have a right adjoint. To conclude, although the opposite category $(\mathbf{DGcoAlg})^\text{op}$ inherits a model structure in the usual way, this model structure does not arise from an application of the transfer theorem.

3.4.2. Hovey criterion. The Quillen adjunction $(\tilde{N}, R^\text{com})$ would be a Quillen equivalence if \tilde{N} reflected weak equivalences between cofibrant objects and if for every fibrant differential graded coalgebra Y , the map $\tilde{N}(R^\text{com}Y)^\text{cof} \rightarrow Y$ was a weak equivalence in **DGcoAlg**. Notice that since every object in **ScoAlg** is cofibrant, the cofibrant replacement $(R^\text{com}Y)^\text{cof}$ can be taken to be $R^\text{com}Y$. Hence, we are reduced to studying the map $\lambda_Y: \tilde{N}R^\text{com}Y \rightarrow Y$ for fibrant coalgebras Y .

Definition 3.11. A fibrant differential graded coalgebra Y satisfies *the Hovey criterion* if λ_Y is a weak equivalence in **DGcoAlg**, that is

$$H_*(\lambda_Y): H_*(\tilde{N}R^\text{com}Y) \cong H_*(Y).$$

We are not able to show that the Hovey criterion holds for an arbitrary fibrant differential graded coalgebra, but we investigate the Hovey criterion for some classes of examples of differential graded coalgebras.

Proposition 3.12. *The following objects in $\mathbf{DGcoAlg}$:*

- (1) *the terminal object $K[0]$,*
- (2) *the cofree coalgebra $S_d(\mathbb{S}^0)$ on the 0-sphere \mathbb{S}^0 ,*
- (3) *the cofree coalgebra $S_d(V)$ on every acyclic vector space V*

are fibrant and satisfy the Hovey's criterion.

Proof. As the functor R^{com} is a right adjoint, it carries the terminal object $K[0]$ to the terminal object of \mathbf{ScoAlg} which is $I(K)$, the simplicial coalgebra with the one dimensional coalgebra K in each degree. Therefore, we obtain

$$H_*(\tilde{N}R^{\text{com}}(K[0])) \cong H_*(\tilde{N}I(K)) \cong H_*(K[0]),$$

hence $K[0]$ satisfies the Hovey criterion.

Recall that every object is fibrant in the model structure on the category \mathbf{DGVct} . Since the functor S_d preserves fibrations, it follows that every cofree coalgebra is fibrant. In particular $S_d(\mathbb{S}^0)$ is fibrant. Now the definition of the functor R^{com} on cofree coalgebras and the Dold-Kan correspondence yields

$$\tilde{N}R^{\text{com}}S_d(\mathbb{S}^0) = \tilde{N}S_s\Gamma(\mathbb{S}^0) \cong \tilde{N}S_s(I(K)).$$

Since the face maps in the simplicial coalgebra $S_s(I(K))$ are all identities on $S(K)$, their alternating sums are either 0 or $\text{id}_{S(K)}$ and the associated complex is:

$$\cdots \longleftarrow 0 \longleftarrow S(K) \xleftarrow{0} S(K) \xleftarrow{\text{id}} S(K) \xleftarrow{0} S(K) \xleftarrow{\text{id}} \cdots$$

Therefore we obtain

$$H_*(\tilde{N}S_s(I(K))) = \begin{cases} 0 & \text{if } * \neq 0 \\ S(K) & \text{if } * = 0 \end{cases}$$

Finally the Hovey criterion for the differential graded coalgebra $S_d(\mathbb{S}^0)$ follows from the fact that $S_d(\mathbb{S}^0)$ is concentrated in degree zero and from the identification of $S_d(\mathbb{S}^0)$ with $S(K)$.

Now let V be an acyclic differential graded vector space. Since the functors S_d , Γ and S_s preserve weak equivalences, we deduce on one hand that $S_d(V)$ is weakly equivalent to $S_d(0) = K[0]$ and on the other hand that

$$\begin{aligned} \tilde{N}R^{\text{com}}S_d(V) &= \tilde{N}S_s\Gamma(V) \simeq \tilde{N}S_s\Gamma(0) \\ &\simeq \tilde{N}S_s(0) = \tilde{N}I(K) = K[0] \end{aligned}$$

Hence, we obtain the required result for acyclic differential graded vector spaces. \square

Notice in particular that the Hovey criterion holds for the cofree coalgebra on every n -disk \mathbb{D}^n .

4. CATEGORIES OF CONNECTED COALGEBRAS

4.1. Connected differential graded coalgebras. In this section we restrict to *connected* differential graded objects. An object V in \mathbf{DGVct} is connected if $V_0 = 0$ and we denote by \mathbf{DGVct}_c the category of connected differential vector spaces. With mild changes, the category \mathbf{DGVct}_c inherits from \mathbf{DGVct} a model category structure. Indeed, \mathbf{DGVct}_c has limits and colimits constructed degreewise as in \mathbf{DGVct} . Moreover, the model category factorization axioms may be performed as in [GS06, Section 1.3], but discarding this time objects such as the 0-sphere \mathbb{S}^0 and the 1-disk $\mathbb{D}^1 \notin \mathbf{DGVct}_c$. In this way, a map f in the category \mathbf{DGVct}_c is a weak equivalence if H_*f is an isomorphism, a cofibration if for each $n \geq 1$, f_n is injective, and a fibration if for each $n \geq 2$, f_n is surjective.

An object C in $\mathbf{DGcoAlg}$ is connected if $C_0 = K$. We denote by $\mathbf{DGcoAlg}_c$ the

category of connected differential graded coalgebras and in the rest of this section we discuss its model category structure.

Now, let V be an object in the category \mathbf{DGVct}_c . Then, the tensor coalgebra on V is defined by $T'_d(V) = \bigoplus_{n \geq 0} V^{\otimes n}$. Since $V_0 = 0$, it follows that $T'_d(V) \cong \prod_{n \geq 0} V^{\otimes n}$. We mention that the structure maps on $T'_d(V)$ are given by

$$\begin{aligned}\Delta_{T'_d(V)}(v_1 \otimes \cdots \otimes v_n) &= \sum_{r=0}^n (v_1 \otimes \cdots \otimes v_r) \otimes (v_{r+1} \otimes \cdots \otimes v_n) \\ \Delta_{T'_d(V)}(1) &= 1 \otimes 1 \\ \epsilon_{T'_d(V)}(v_1 \otimes \cdots \otimes v_n) &= 0 \text{ for } n \geq 1 \text{ and } \epsilon_{T'_d(V)}(1) = 1.\end{aligned}$$

Definition 4.1. [HMS74, Section II.2]. Let C be a connected differential graded coalgebra. Then, the functor $I'_d: \mathbf{DGcoAlg}_c \rightarrow \mathbf{DGVct}_c$ is defined by

$$I'_d(C) = C/K[0].$$

In other words, the differential graded vector space $I'_d(C)_n$ is C_n for $n \geq 1$ and 0 otherwise.

Lemma 4.2. [HMS74, Section II.2]. *The tensor coalgebra functor $T'_d: \mathbf{DGVct}_c \rightarrow \mathbf{DGcoAlg}_c$ is right adjoint to the functor I'_d .*

Proposition 4.3. *The category of connected differential graded coalgebras is complete and cocomplete.*

Proof. We start with limits. A terminal object in $\mathbf{DGcoAlg}_c$ is given by $K[0]$. For other limits, we recall from [GG99, Proposition 1.7] that there is an anti-equivalence between the category of differential graded coalgebras and the category of profinite differential graded algebras. Since by the appendix in Section 5, the category of connected differential graded algebras is complete and cocomplete, we derive limits for $\mathbf{DGcoAlg}_c$ by applying the steps given in [GG99, Proof of Proposition 1.8]. We mention that the usual tensor product \otimes of differential graded coalgebras is not the categorical product in $\mathbf{DGcoAlg}_c$ since we do not assume cocommutativity.

Next, we investigate colimits. We refer to [Nei78, Section 1] where similar constructions appear for cocommutative coalgebras. Since, as a left adjoint, the functor I'_d must preserve initial objects, we deduce that $K[0]$ is initial in $\mathbf{DGcoAlg}_c$. Let $f, g: C \rightarrow D$ be two maps in $\mathbf{DGcoAlg}_c$. Then, their coequalizer is given by $D/\text{im}(f - g)$. This quotient is constructed degreewise and each of its homogeneous parts is in fact a coalgebra by [Swe69, Proposition 1.4.8]. Finally, if C and D are two objects in $\mathbf{DGcoAlg}_c$, we may form the maps

$$K[0] \xrightarrow{\varphi_C} C \xrightarrow{i_C} C \oplus D \quad \text{and} \quad K[0] \xrightarrow{\varphi_D} D \xrightarrow{i_D} C \oplus D.$$

Then the coproduct of C and D in $\mathbf{DGcoAlg}_c$ is given by

$$C \sqcup D = C \oplus D/\text{im}(i_C \circ \varphi_C - i_D \circ \varphi_D).$$

Notice that the direct sum is the coproduct of the underlying differential graded vector spaces and that the quotient guarantees the required connectedness condition. \square

After these categorical facts, we endow the category of connected differential graded coalgebras with a model structure which is defined exactly as in Section 3.2.2. We essentially concentrate on the model category factorization axioms.

Lemma 4.4. *Let V be an acyclic ($H_* V = 0$) connected differential graded vector space and C , a connected differential graded coalgebra. Then the projection*

$$C \sqcap T'_d(V) \longrightarrow C$$

is an acyclic fibration.

Proof. By using [GG99, Proof of Lemma 1.12], we obtain the isomorphisms

$$[C \sqcap T'_d(V)]^* \cong C^* \sqcup [T'_d(V)]^* \cong C^* \sqcup T_d(V^*) \cong T_{C^*}(C^* \otimes V^* \otimes C^*)$$

where T_{C^*} denotes the tensor C^* -algebra functor. The cohomological Künneth formula yields

$$H^*(T_{C^*}(C^* \otimes V^* \otimes C^*)) \cong H^*(C^*).$$

The required homology isomorphism $H_*(C \sqcap T'_d(V)) \cong H_*C$ comes from applying the cohomological universal coefficient theorem.

It remains to prove that each projection is a fibration. To see this, we consider the diagram

$$\begin{array}{ccc} A & \longrightarrow & C \sqcap T'_d(V) \\ i \downarrow & & \downarrow \\ B & \longrightarrow & C \end{array}$$

where i is a cofibration in **DGcoAlg**_c. By adjointness, this amounts to considering the diagram

$$\begin{array}{ccc} I'_d(A) & \longrightarrow & V \\ I'_d(i) \downarrow & & \downarrow \\ I'_d(B) & \longrightarrow & 0 \end{array}$$

in **DGVct**_c. Since $I'_d(i)$ is a cofibration and $V \rightarrow 0$ is an acyclic fibration in the model category of **DGVct**_c, a lift $I'_d(B) \rightarrow V$ exists. \square

Proposition 4.5. *Let $f: C \rightarrow D$ be a morphism in **DGcoAlg**_c. Then, the morphism f can be factored as*

$$C \xrightarrow{i} X \xrightarrow{p} D$$

with i a cofibration and p an acyclic fibration.

Proof. Let us consider the object $X = D \sqcap T'_d(\text{cone}(I'_d(C))) \in \mathbf{DGcoAlg}_c$. Since $\text{cone}(I'_d(C)) \in \mathbf{DGVct}_c$ is acyclic, Lemma 4.4 implies that the projection $p: X \rightarrow D$ is an acyclic fibration.

Next, the following diagram shows how to obtain the coalgebra map i

$$\begin{array}{ccccc} & & f & & \\ & \swarrow & & \searrow & \\ C & \xrightarrow{\quad i \quad} & D \sqcap T'_d(\text{cone}(I'_d(C))) & \xrightarrow{\quad p \quad} & D \\ & \searrow & & \swarrow & \\ & & j & & \pi \\ & & & & T'_d(\text{cone}(I'_d(C))). \end{array}$$

Indeed, j comes from the composition $C \rightarrow I'_d(C) \rightarrow \text{cone}(I'_d(C))$ and the universal property of the tensor coalgebra functor T'_d . Finally, the map i is a cofibration since

$$I'_d(j): I'_d(C) \rightarrow T'_d(\text{cone}(I'_d(C))) = K[0] \oplus \text{cone}(I'_d(C)) \oplus \cdots$$

is a canonical injection map. \square

Proposition 4.6. *Any morphism $f: C \rightarrow D$ in $\mathbf{DGcoAlg}_c$ can be factored as*

$$C \xrightarrow{i} G \xrightarrow{p} D$$

with i an acyclic cofibration and p a fibration.

Sketch of proof. We exploit arguments in [Smi11, Definition 4.14, Lemma 4.16] for our purpose. First, factor f as $C \xrightarrow{i_0} D \sqcap T'_d(I'_d C) \xrightarrow{p_0} C$ where the map i_0 is a canonical injection, hence a cofibration in $\mathbf{DGcoAlg}_c$ and p_0 a fibration. Then, construct the maps $(i_n)_{n \geq 1}$ and $(p_n)_{n \geq 1}$ as displayed in the diagram

$$\begin{array}{ccccc} C & \xrightarrow{i_0} & G(0) = D \sqcap T'_d(I'_d C) & \xrightarrow{p_0} & D \\ & \searrow i_1 & \downarrow p_1 & & \\ & \searrow i_n & \downarrow & & \\ & \searrow i_{n+1} & \downarrow & & \\ & \searrow i_\infty & \downarrow p_{n+1} & & \\ & & G(n) & & \\ & & \downarrow & & \\ & & G(n+1) & & \\ & & \downarrow & & \\ & & \lim_n G(n) & & \end{array}$$

in which

- (1) for $n \geq 0$, the objects $G(n+1)$ are pullbacks of

$$G(n) \longrightarrow T'_d(I'_d H(n)) \longleftarrow T'_d(s^{-1}\text{cone}(I'_d H(n)))$$

where $H(n)$ are pushouts of $G(n) \longleftarrow C \xrightarrow{\epsilon_C} K[0]$ and s^{-1} the desuspension

- (2) the maps $(p_n)_{n \geq 1}$ form a tower of fibrations. In fact, as pullbacks, they are built out of \mathbf{DGVct}_c fibrations $s^{-1}\text{cone}(I'_d H(n)) \longrightarrow I'_d H(n)$.
- (3) the maps $(i_n)_{n \geq 1}$ are cofibrations since by induction the compositions $p_n \circ i_n$ are injections.
- (4) the object $\lim_n G(n)$ is weak equivalent to C since the object

$$\left(\lim_n G(n)\right) / i_\infty(C)$$

is acyclic by similar arguments as in [Smi11, Proof of Lemma 4.16].

Therefore, setting $i = i_\infty$ and $p = p_0 \circ p_1 \circ \dots$, gives the required factorization. \square

Lemma 4.7. *Let C be a fibrant connected differential graded coalgebra and*

$$C \xrightarrow{i} G \xrightarrow{p} K[0]$$

be the factorization of $\epsilon_C: C \rightarrow K[0]$ as in the previous lemma. Then, C is a retract of G . Moreover, G is a cofree connected differential graded coalgebra.

Proof. Since C is fibrant, the counit ϵ_C is a fibration. It follows that a lift exists in the diagram

$$\begin{array}{ccc} C & \xlongequal{\quad} & C \\ i \downarrow & \nearrow \pi & \downarrow \epsilon_C \\ G & \xrightarrow{p} & K[0] \end{array}$$

and therefore that C is a retract of G .

For the second statement, note first that $G(0)$ is cofree in $\mathbf{DGcoAlg}_c$ since

$$G(0) = T'_d(I_d C) \sqcap K[0] = T'_d(I_d C) \sqcap T'_d(0) \cong T'_d(I'_d C \times 0) \cong T'_d(I'_d C).$$

As a right adjoint functor, T'_d commutes with inverse limits. Hence, we may recursively deduce that for $n \geq 1$ the objects $G(n)$ are cofree in $\mathbf{DGcoAlg}_c$ and finally that $G = \lim_n G(n)$ is cofree as required. \square

4.2. Connected simplicial coalgebras. This section is the simplicial counterpart of the previous one. An object V in \mathbf{SVct} is connected if $V_0 = 0$. A connected simplicial coalgebra C is an object in \mathbf{ScoAlg} with $C_0 = K$. We denote by \mathbf{SVct}_c and \mathbf{ScoAlg}_c the categories of connected simplicial vector spaces and coalgebras.

Lemma 4.8. *Let C be a connected simplicial coalgebra. Then, the constant simplicial object $I(K)$ splits off the object C .*

Proof. In the category \mathbf{ScoAlg}_c , the object $I(K)$ is both initial and terminal as proven in Proposition 4.11 below. Consequently, the canonical map $i: I(K) \rightarrow C$ is an injection and therefore $I(K)$ splits off C . \square

We next consider $T': \mathbf{Vct} \rightarrow \mathbf{coAlg}$ the tensor coalgebra functor constructed in [Swe69, Theorem 6.4.1]. The simplicial tensor coalgebra $T'_s: \mathbf{SVct}_c \rightarrow \mathbf{ScoAlg}_c$ is defined as a degreewise prolongation of the tensor coalgebra functor T' .

Definition 4.9. Let C be a connected simplicial coalgebra. Then, the functor $I'_s: \mathbf{ScoAlg}_c \rightarrow \mathbf{SVct}_c$ is defined by $I'_s(C) = C/I(K)$. Notice that this definition makes sense because of Lemma 4.8.

Proposition 4.10. *Let V be a connected simplicial vector space and 1_K denotes a generator of $I(K)$. If C is a connected simplicial coalgebra, then there is a bijection from the set of simplicial coalgebra maps $C \rightarrow T'_s(V)$ to the set of connected simplicial vector spaces maps $I'_s(C) \rightarrow V$ such that $u(1_K) = 0$.*

Proof. In the following,

$$\begin{aligned} \mathbf{ScoAlg}_c(C, T'_s(V)) &\cong \mathbf{SVct}_c(C, V) \\ &\cong \mathbf{SVct}_c(C/I(K), V) \\ &= \mathbf{ScoAlg}_c(I'_s(C), V) \end{aligned}$$

the first bijection comes from the fact that T'_s is right adjoint to the forgetful functor from simplicial coalgebras to simplicial vector spaces. The second bijection follows from the assumption $u(1_K) = 0$ which implies that $I(K) \subseteq \ker u$. \square

Proposition 4.11. *The category of connected simplicial coalgebras is complete and cocomplete.*

Proof. For limits in \mathbf{ScoAlg}_c , it suffices to extend degreewise the construction by [Ago11, Theorem 1.1] for the category of coalgebras over fields. Hence, a terminal object in \mathbf{ScoAlg}_c is given by $I(K)$, the constant simplicial coalgebra. Notice that, since the field K is a terminal object in \mathbf{coAlg} , the product $K \sqcap K$ is isomorphic to K . This ensures that the products $C \sqcap D$ of objects in \mathbf{ScoAlg}_c is again connected since $(C \sqcap D)_0 = C_0 \sqcap D_0 = K \sqcap K = K$.

Colimits in \mathbf{ScoAlg}_c are formed in the same way as for $\mathbf{DGcoAlg}_c$. In this way, an initial object is given by $I(K)$. If $f, g: C \rightarrow D$ are two maps in \mathbf{ScoAlg}_c , their coequalizer is given by $D/\text{im}(f - g)$. Finally, If C and D are two objects in \mathbf{ScoAlg}_c , we may form the maps

$$I(K) \xrightarrow{\varphi_C} C \xrightarrow{i_C} C \oplus D \quad \text{and} \quad I(K) \xrightarrow{\varphi_D} D \xrightarrow{i_D} C \oplus D.$$

Then the coproduct of C and D in \mathbf{ScoAlg}_c is given by

$$C \sqcup D = C \oplus D / \text{im}(i_C \circ \varphi_C - i_D \circ \varphi_D).$$

Notice that the direct sum is taken degreewise and that the quotient guarantees the connectedness condition. \square

With the above definition, the category \mathbf{ScoAlg}_c is endowed with a model category structure exactly as in [Goe95, Section 3].

4.3. A Quillen equivalence for connected coalgebras. In this section, we improve the Quillen adjunction $(\tilde{N}, R^{\text{com}})$ to a Quillen equivalence. Recall that we were not able to check the Hovey's criterion for arbitrary fibrant differential graded coalgebras. However, connectedness is a condition that guarantees such a criterion which yields a Quillen equivalence.

Lemma 4.12. *There is an equivalence between the category of connected simplicial vector spaces and the category of connected differential graded vector spaces.*

Proof. We notice that the restriction of the normalization functor $N: \mathbf{SVct}_c \rightarrow \mathbf{DGVct}_c$ is full and faithful since it is induced by the Dold-Kan equivalence. Moreover, if V an object in \mathbf{DGVct}_c , we may find an object W in \mathbf{SVct}_c such that $NW \cong V$. Indeed, since $\Gamma(V)_0 = V_0 = 0$, it follows that $\Gamma(V) \in \mathbf{SVct}_c$ and setting $W = \Gamma(V)$ satisfies the required condition. Therefore, by [Par70, Section 2.1, Proposition 3], we deduce that the restriction of the normalization functor N induces an equivalence between connected vector spaces categories with an inverse given by the restriction of Γ . \square

The following result is dual to [Qui69, Part I, Proposition 4.5].

Lemma 4.13. *Let V be a differential graded vector space. Then the following maps*

$$H_*(\tilde{N}T'_s\Gamma(V)) \longrightarrow H_*(T'_d(V)) \longrightarrow T'_dH_*(V)$$

of graded coalgebras are isomorphisms.

Proof. These maps are obtained by using the universal properties of the respective tensor coalgebras. Then, as vector spaces, the tensor coalgebra is the same as the tensor algebra, hence we obtain the isomorphisms by applying Künneth and Eilenberg-Zilber theorems. \square

Lemma 4.14. *If C is a cofree differential graded coalgebra, then the map*

$$\tilde{N}R^{\text{com}}C \longrightarrow C$$

is a weak equivalence.

Proof. Since $C \cong T'_d(V)$ one has

$$H_*(\tilde{N}R^{\text{com}}C) \cong H_*(\tilde{N}T'_s\Gamma(V)) \cong H_*(T'_d(V)) \cong H_*(C)$$

and hence the required result. \square

Theorem 4.15. *If C is a fibrant connected differential graded coalgebra, then the map*

$$\tilde{N}R^{\text{com}}C \longrightarrow C$$

is a weak equivalence. Hence, there is a Quillen equivalence between the category of connected differential graded coalgebras and the category of connected simplicial coalgebras.

Proof. Recall in Lemma 4.7 that C is a retract of a cofree coalgebra G , which may be written as $T'_d(V)$ for some $V \in \mathbf{DGVct}_c$. Applying successively the functors R^{com} , \tilde{N} and H_* to the retract map $C \rightarrow T'_d(V) \rightarrow C$ we obtain the diagram

$$\begin{array}{ccccc} H_*(\tilde{N}R^{\text{com}}C) & \longrightarrow & H_*(\tilde{N}T'_s\Gamma(V)) & \longrightarrow & H_*(\tilde{N}R^{\text{com}}C) \\ \downarrow & & \downarrow \cong & & \downarrow \\ H_*C & \longrightarrow & H_*(T'_d(V)) & \longrightarrow & H_*C \end{array}$$

of homology morphisms. Since, by Lemma 4.13, $H_*(\tilde{N}T'_s\Gamma(V)) \cong H_*(T'_d(V))$, we deduce that $H_*(\tilde{N}R^{\text{com}}C) \cong H_*C$ with help of [DS95, Lemma 2.7]. Then, the Quillen equivalence follows from [Hov99, Corollary 1.3.16]. \square

5. APPENDIX ON CONNECTED DIFFERENTIAL GRADED ALGEBRAS

In this appendix, we consider the category of connected differential graded algebras. This category is somehow dual to the category of connected differential graded coalgebra. We refer to [GG99, Proposition 1.7] for the precise statement. Its main interest here, is that a particular dual of its limits is used to construct colimits for the category $\mathbf{DGcoAlg}_c$.

Definition 5.1. A *connected* differential graded algebra A is a differential graded vector space equipped with an associative, graded multiplication $M: A \otimes A \rightarrow A$ and a unit $\mu: K[0] \rightarrow A$ with $A_0 = K$.

We denote by $\mathbf{DGA}_{\text{Alg}}$ the category of connected differential graded algebras.

Definition 5.2. Let A be an object in the category $\mathbf{DGA}_{\text{Alg}}$. The isomorphism $\mu|_{A_0}: K \rightarrow A_0$ induces a map $\gamma_A: A \rightarrow K[0]$ and we define a functor $I_d: \mathbf{DGA}_{\text{Alg}} \rightarrow \mathbf{DGVct}_c$ by $I_d(A) = \ker \gamma_A$.

Lemma 5.3. *The tensor algebra functor $T_d: \mathbf{DGVct}_c \rightarrow \mathbf{DGA}_{\text{Alg}}$ is left adjoint to the functor I_d .*

Proposition 5.4. *The category of connected differential graded algebras is complete and cocomplete.*

Proof. Constructions of limits are well-known in the category of differential graded algebras. Because of connectedness, some refinements have to be performed. A terminal object in $\mathbf{DGA}_{\text{Alg}}$ is given by $K[0]$. If $f, g: A \rightarrow B$ are two maps in $\mathbf{DGA}_{\text{Alg}}$, their equalizer is given by $\ker(f - g)$. Now let A and B be objects in $\mathbf{DGA}_{\text{Alg}}$. We may form the maps

$$A \times B \xrightarrow{\pi_A} A \xrightarrow{\gamma_A} K[0] \quad \text{and} \quad A \times B \xrightarrow{\pi_B} B \xrightarrow{\gamma_B} K[0].$$

Then the product of A and B in $\mathbf{DGA}_{\text{Alg}}$ is given by

$$A \sqcap B = \ker(\gamma_A \circ \pi_A - \gamma_B \circ \pi_B).$$

For colimits, we first notice that an initial object in $\mathbf{DGA}_{\text{Alg}}$ is given by $K[0]$. Then the coequalizer of two maps $f, g: A \rightarrow B$ is given by

$$B / \langle f(a) - g(a), a \in A \rangle$$

where $\langle f(a) - g(a), a \in A \rangle$ denotes the ideal generated by $f(a) - g(a)$ for $a \in A$. In [Jar97], the coproduct of two differential graded non-commutative algebras A and B is given by factoring out from the tensor algebra $T_d(A \otimes B)$ the ideal \mathcal{I} which is generated by elements of the form

$$\begin{aligned} (a_1 \otimes b_1) \otimes (1 \otimes b_2) - a_1 \otimes b_1 b_2, \\ (a_1 \otimes 1) \otimes (a_2 \otimes b_2) - a_1 a_2 \otimes b_2. \end{aligned}$$

However, the resulting object need not to be connected even if A and B are connected. To get around this problem, we define the coproduct of two objects in $\mathbf{DGA}_{\mathbf{c}}$ as

$$A \sqcup B = T_d(\ker \gamma_A \otimes \ker \gamma_B) / \mathcal{I}.$$

Since $\ker \gamma_A$ and $\ker \gamma_B$ are connected differential graded vector spaces, they will not contribute to the degree 0 part of the tensor algebra $T_d(\ker \gamma_A \otimes \ker \gamma_B)$. In this way, we will have $(A \sqcup B)_0 = K$ and therefore $A \sqcup B \in \mathbf{DGA}_{\mathbf{c}}$. \square

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